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Matrix model for de Sitter

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Abstract: Based on a heuristic boost argument, we propose that the 4 dimensional de Sitter space can be described by a spherical Chern-Simons matrix model near the cosmological horizon, or models generalizing this simple choice. The dimension of the Hilbert space is naturally finite. We also make some comments on possible realization of holography in this approach, and on possible relation to the conformal field theory approach.

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1. Introduction

The problem of formulating a sensible theory of quantum gravity on de Sitter space has attracted increasingly more attention [1]–[4]. Some earlier attempts include [5]. The cause of these interests seems twofold. For one thing, string theorists have gained much control in the past few years over string theory when there is enough supersymmetry, the space-time background in this case is either Minkowski or anti-de Sitter, and there is a proposed nonperturbative formulation in each case, namely Matrix Theory or Maldacena's CFT description. As soon as supersymmetry is completely broken, we have not yet succeeded in finding a Minkowski background, and most likely, as proposed in [1], we will end up with a background with a positive cosmological constant. Several theoretical difficulties inherent to such a background have been discussed in [1, 2, 3]. Among these, the problem of formulating observables is particularly challenging. On the other hand, several observational results [7, 8] point to the possibility that we are living in an accelerating universe with considerable amount of dark energy, This has induced disquieting concerns in the string community that such reality might contradict our current understanding of string theory [9].

One encounters two seemingly insurmountable difficulties in trying to embed de Sitter space in string theory. First, how to generate a vanishingly small cosmological constant? [10]. Second, even if this can be done, how to describe physics in a de Sitter background? These two problems are likely to be deeply intertwined. In this note we shall follow the philosophy that we may still construct a sensible theory given a de Sitter background, temporarily forgetting the hard problem of generating a small cosmological constant.

One of the first things coming to sight when one tries to formulate a theory in the de Sitter background is that the vacuum structure of de Sitter ought to be more complicated than that of either Minkowski or anti-de Sitter, since the entropy of the background as seen by a co-moving observer is nonvanishing ($> 10^{120}$). There is a tiny Hawking temperature. However, one can not attribute the huge amount of entropy to thermal fluctuations at the scale of the Hawking temperature, since on general grounds of thermodynamics the entropy of this kind is of order 1 only. Also, the thermal fluctuation of this kind in principle is not observable at all, since the wave length is the size of the horizon $R \sim H^{-1}$, while a co-moving observer has to be well-localized. In fact, even the entropy due to baryonic matter is much smaller than the Bekenstein-Hawking entropy.

We are thus forced to propose that the entropy, or the large Hilbert space, is mainly composed of unusual degrees of freedom which we shall call wee-degrees. Their dynamics must be essentially different from ordinary particles of massless or massive, yet they are coupled to ordinary matter. For having a consistent theory, it is better to postulate that ordinary matter is collective excitations of wee-degrees. This requirement is supported by the validity of the so-called D-bound [11], which states that the entropy of any matter system within the cosmological horizon plus the Bekenstein Hawking entropy should be no larger than the Bekenstein-Hawking entropy after the matter system disappears from the horizon. In terms of our picture, this is just $S_{\rm matter} + S_{\rm wee}$ increases during the course when the matter system falls out the horizon and is bounded by $S_{\rm wee}$, the entropy of the resulting empty de Sitter space inside the final horizon. A stronger bound, the holographic bound, also implies that the wee-degrees of freedom should account for the whole entropy enclosed in the cosmological horizon.

There are at least two criteria for an acceptable candidate theory of wee-degrees: 1. The theory must account for the large entropy; 2. When a normal matter degree is coupled to the wee-degrees, the effective equation of motion reproduced must be identical to one in de Sitter space. We will take a step toward meeting the first criterion. In the next section we will present a boost argument borrowed from the construction of matrix theory [12], we argue that a matrix model of first order in time derivatives is a reasonable candidate. We then proceed in section 3 to propose a simplest possible matrix model, the Chern-Simons matrix model on the stretched horizon, to describe the de Sitter space in 4 dimensions. De Sitter boost properties and a conformal light-cone frame are discussed in section 5. In section 6 we argue that the light-cone coordinate is the best choice of time in the matrix model, and it is related to dilation generator on the infinite past Euclidean surface. We conclude this paper in section 7 with some discussions.

2. Intuitive considerations

Suppose we study a system immersed in an environment of constant energy density ρ in a volume V, and this environment is Lorentz invariant. When the system is at static, the total momentum $P = 0$, the total energy is $M + \rho V$. Boost the system, there will a net momentum P coming only from the system, the environment does not contribute. Now there are two possible on-shell conditions, depending on which view-point we are going to

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take. The first is that the system is completely decoupled from the environment, so the energy of the system and the momentum satisfy

$$E^2 = P^2 + m^2 \,. \tag{2.1}$$

We do not get anything interesting from this relation. Another possibility is to assume that the system in fact is kind of collective excitation of the environment, so the total energy and total momentum satisfy a single on-shell condition

$$(E + \rho V)^2 = (M + \rho V)^2 + P^2 \,. \tag{2.2}$$

If the energy in the environment is much greater than M and P, we obtain, approximately

$$E = M + \frac{P^2}{2(PV + M)} + O(P^4) \,, \tag{2.3}$$

in this way, the system behaves as a massive system with a large mass $\rho V + M \sim \rho V$. This intuition is quite right, since this implies that it is extremely hard to excite the environment whose prototype is the non-dynamic vacuum energy, or the cosmological constant.

To imitate matrix theory, we go to the limit in which the longitudinal momentum P is either at the same scale of ρV or even much greater than ρV. Denote $P_{\text{eff}}^2 = (\rho V)^2 + P^2$, then

$$E + \rho V - P_{\text{eff}} = \frac{M^2}{2P_{\text{eff}}} + \frac{\rho V M}{P_{\text{eff}}} \,. \tag{2.4}$$

Since $M \ll \rho V$, the second term dominates the first term on the r.h.s. of the above equation. If one names $E + \rho V - P_{\text{eff}} = P_-$, it is analogous to the light-cone energy.

If something similar to the situation in matrix theory holds, then one may identify the first term with a hamiltonian similar to that in matrix theory, that is

$$\frac{M^2}{2P_{\text{eff}}} \sim \text{tr}\, \frac{1}{2R} (\partial_t X^i)^2 + V(X) \,, \tag{2.5}$$

where X^i are matrices corresponding to the transverse coordinates, R is some IR cut-off in the longitudinal direction, which may be identified with the horizon size. Let us push this analogue a little further, then the second term, the more important one, is to be identified with

$$\frac{\rho V M}{P_{\text{eff}}} = \left(\frac{\sqrt{2}\rho V}{\sqrt{P_{\text{eff}}}} \right) \left(\frac{M}{\sqrt{2P_{\text{eff}}}} \right) \sim \frac{\sqrt{2}\rho V}{\sqrt{P_{\text{eff}}}} \left(\text{tr}\, \frac{1}{2R} (\partial_t X^i)^2 + V(X) \right)^{1/2} \,. \tag{2.6}$$

The square root in the last equality is not an analytic function, so itself does not make sense as a candidate for the dominant term in the hamiltonian. We however take the square root as suggesting that the hamiltonian should be of first order in derivatives of X^i. In the supersymmetric matrix theory, the hamiltonian is the square of a super charge. This super charge can not be used as a candidate for our hamiltonian, since in the first place there is no supersymmetry in a de Sitter space, and secondly the super charge is fermionic so it can not serve as a hamiltonian.

In any case, we are tempted by the possibility that in a certain frame, the hamiltonian of wee-degrees is that of matrices and is at most of the first order in time derivatives of these

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matrices. Suppose the action principle is applicable here, then the hamiltonian should be independent of the time derivatives at all, since the action is at most of the first order in the time derivatives. This statement is not modified when we work in the de Sitter space with judiciously chosen coordinates, as we shall see in the next sections.

For now let us simply postulate that in a certain frame the wee-degrees are described by matrices with an action first order in time derivatives. The general form is

$$S = \int dt \, \mathrm{tr} \left(P^i(X) \partial_t X^i - V(X) \right) . \tag{2.7}$$

The canonical conjugate of X^i is the transpose of $P^i(X)$, a function of X only. The hamiltonian is equal to $\mathrm{tr}\, V(X)$. Assume for hermitean matrices, the hamiltonian assumes its lowest value for mutually commuting matrices, as in matrix theory. Thus in the ground state only the diagonal entries of X^i are physical degrees of freedom.

De Sitter space is invariant under boost, thus it is not possible to assign a longitudinal momentum to a diagonal degree of freedom, since unlike in matrix theory in Minkowski space, we are trying to describe an empty de Sitter space and possible excitations within. Thus the rank of matrices X^i is not to be interpreted as the total longitudinal momentum or a certain charge.

We shall argue later that X^i should be treated either as the transverse coordinates in the steady-state metric of de Sitter, or the spherical coordinates in the static metric. In either case there are two coordinates, and the eigenvalues are subject roughly to the condition

$$(X_a^1)^2 + (X_a^2)^2 \leq R^2 , \tag{2.8}$$

where X_a^i is the a-th eigenvalue, and R is the horizon radius. Let the corresponding eigenvalue of P^i be P_a^i, it is not independent of X_a^i in general. So the phase space of the a-th eigenvalues of (X^i) is two dimensional. To have a compatible Poisson structure, we require

$$\frac{\partial P_a^1}{\partial X_a^2} = -\frac{\partial P_a^2}{\partial X_a^1} , \tag{2.9}$$

so the Poisson bracket between X_a^1 and X_a^2 is

$$\{X_a^1, X_a^2\} = \left(\frac{\partial P_a^1}{\partial X_a^2} \right)^{-1} . \tag{2.10}$$

The dimension of the Hilbert space of the a-th single eigenvalues is, on the semiclassical level

$$\dim H_a = \frac{1}{2\pi} \int \left| \frac{\partial P_a^1}{\partial X_a^2} \right| dX_a^1 dX_a^2 . \tag{2.11}$$

This integral is finite for the integration range of X_a^i is finite.

The simplest choice of P^i is $l^{-2} \epsilon^{ij} X^j$, where l is a length scale. This is just the case when the kinetic term in the action is Chern-Simons like. To avoid double counting of the phase space, the kinetic term is actually $\frac{1}{2} \int dt \, \mathrm{tr}\, P^i \dot{X}^i$. The dimension of the Hilbert space of the a-th eigenvalues in this case is

$$\dim H_a = h = \frac{R^2}{2\pi l^2} . \tag{2.12}$$

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To compute the dimension of the whole Hilbert space, we now need to assign certain statistics to the eigen-values. Let N is the rank of matrices X^i. If all eigen-values are distinguishable, the whole Hilbert space of the lowest energy is

$$H = (H_a)^N \,. \tag{2.13}$$

If, due to gauge symmetry, the eigenvalues are treated as bosons, then the whole Hilbert space is

$$H = \mathrm{Sym}(H_a)^N \,, \tag{2.14}$$

where we need to symmetrize the tensor product. In the first case, the dimension of the total Hilbert space is

$$\dim H = (\dim H_a)^N = \exp\left(N \ln \frac{R^2}{(2\pi l)^2}\right). \tag{2.15}$$

To equal the exponent to the Bekenstein Hawking entropy, we must have

$$N \sim S \sim \frac{R^2}{l_p^2} \,, \tag{2.16}$$

where we have ignored the factor $\ln[R^2/(2\pi l)^2]$, since it is not a large number. Even for $l = l_p$, in reality this factor is about $120 \ln 10$. We conclude that in the "infinite statistics" case, the rank of the matrices must be the order of the entropy.[1]

If eigenvalues obey Bose-Einstein statistics, the dimension of the total Hilbert space is

$$\dim H = \frac{(N+h-1)!}{N!(h-1)!} \,. \tag{2.17}$$

We shall always assume both N and h be large numbers. Using the Sterling formula,

$$S = \ln \dim H = N \ln\left(1 + \frac{h}{N}\right) + h \ln\left(1 + \frac{N}{h}\right). \tag{2.18}$$

If $N \ll h$, then the second term is about N, so the first term dominates, this requires that $S \sim N$. If $N \gg h$, the second term dominates, and $S \sim h$. In either case, the entropy is at the same order of the smaller number of (N, h). To us, the most likely case is $N \sim h \sim S$. For the Chern-Simons model, we demand $l \sim l_p$, and $N \sim R^2/l_p^2$. We shall argue in later sections that this is a reasonable choice.

To conclude, we see that within in the framework of the first order action, it is possible to account for the cosmological entropy whether or not the eigen-values are treated as bosons. In both cases, the rank N must be not too smaller than the entropy. It is nonetheless important that the dimension of the Hilbert space of the single eigenvalue is finite, otherwise the dimension of the total Hilbert is infinite.

[1] We caution here that sometimes in the literature this statistics is called Boltzmann statistics, this however is a misnomer, since the traditional Boltzmann statistics treats particles as identical, but assumes that the particle number is much smaller than the the number of single particle states, so it is the high temperature limit of both Bose-Einstein statistics and Fermi-Dirac statistics.

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3. The spherical matrix Chern-Simons model

De Sitter space of D dimensions can be regarded as a hyper surface in a $D+1$ dimensional Minkowski space with the constraint

$$-X_0^2 + \sum X_i^2 = R^2 \,. \tag{3.1}$$

The de Sitter metric is the induced metric. There are three well-known coordinate systems. The global coordinates are

$$X_0 = R\sinh\tau, \quad X_i = R\cosh\tau x_i \,, \tag{3.2}$$

where x_i form a D dimensional unit vector thus parameterize the unit sphere S^{D-1}. The metric is written as

$$ds^2 = R^2\left(-d\tau^2 + \cosh^2\tau d\Omega_{D-1}^2\right). \tag{3.3}$$

Another set of coordinates is

$$\begin{aligned} t &= R\ln\frac{X_0+X_D}{R}\,, \\ x_i &= \frac{RX_i}{X_0+X_D}\,, \qquad i<D\,, \end{aligned} \tag{3.4}$$

the definition of t demands $X_0 + X_D > 0$ thus these coordinates cover only half of the de Sitter space. The metric in terms of them is the one for the steady-state universe

$$ds^2 = -dt^2 + e^{2t/R}dx_i^2 \,. \tag{3.5}$$

We shall discuss some properties of Killing vectors of this metric in section 5.

Finally, the coordinates we will mostly use in this section are the ones for the static metric

$$t = -\frac{R}{2}\ln\frac{X_0^2}{(X_0+X_D)^2}\,, \qquad x_i = X_i, i<D\,. \tag{3.6}$$

The metric reads

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)dt^2 + \left(1-\frac{r^2}{R^2}\right)^{-1}dr^2 + r^2 d\Omega_{D-2}^2 \,. \tag{3.7}$$

In the following we will focus on the 4 dimensional de Sitter space, though the geometric discussions apply equally to other de Sitter spaces. An observer sitting at $r=0$ is a geodesic observer, this corresponds to an observer in the steady-state universe at $x_i = 0$. It takes an infinite amount of time for a light signal originating from $r=0$ to reach the horizon $r=R$, in the static coordinates. In the steady-state universe, the light reaches horizon in finite time, this time is the same as recorded by the observer at the origin, so there seems to be a contradiction. However, it takes infinite amount of time for this light to return to the origin, so the observer sees that the light never gets to the horizon in finite time of his clock. The meaning of the horizon therefore is invariant.

As in many works on black holes [13], we will here advocate a stretched horizon picture. This picture is perceived by the observer at the origin, and again this picture is invariant

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with respect to changing coordinate system. It is however convenient to work with the static coordinates. In a normal approach, the stretched horizon is supposed to be located a proper Planck distance away from the real horizon. Let r_0 be the location of the stretched horizon, its proper distance to $r = R$ is

$$d = \int_{r_0}^{R} dr \left(1 - \frac{r^2}{R^2}\right)^{-1/2} = R \sin^{-1} \sqrt{1 - \frac{r_0^2}{R^2}} \sim R\sqrt{1 - \frac{r_0^2}{R^2}}\,. \tag{3.8}$$

If $d \sim l_p$, then the red-shift factor

$$\sqrt{1 - \frac{r_0^2}{R^2}}$$

is about l_p/R. For a macroscopic de Sitter, this is a tiny number.

Now for a stationary observer near the horizon, all outgoing matter will get a huge boost, and the boost factor is given by $\exp(t/R)$. This can be obtained by going to the tortoise radial coordinate

$$r^* = \frac{R}{2} \ln \frac{1 + r/R}{1 - r/R}\,, \tag{3.9}$$

with a metric

$$ds^2 = \frac{4e^{2r^*/R}}{(e^{2r^*/R} + 1)^2}(-dt^2 + dr^{*2}) + \cdots\,. \tag{3.10}$$

The trajectory of a particle is roughly $t = r^*$, so the red-shift factor is $\exp(-r^*/R) = \exp(-t/R)$, a particle of energy ϵ_0 at the origin now has an energy $\exp(t/R)\epsilon_0$. If we do not introduce a cut-off on t, we eventually put the particle in the infinite momentum frame. Thus, our intuitive consideration in the previous section applies, and we end up with, if correct, a system described by a first order action. Notice that the fictitious observers hovering on the horizon can communicate with the observer at the origin, so for them there is a single theory.

In the stretched horizon picture [14], the boost is finite, and the boost factor is R/l_p. A wave length $1/R$ is boosted to a wave length $1/l_p$, a planckian particle. It is therefore natural to suppose that the relevant length scale in the Chern-Simons type action is set by l_p, as we already did in the last section. In this case, our intuitive boost argument is to be taken merely as an argument, since the boost can not be made infinite. Locally, we now have a flat Chern-Simons matrix model, globally, the Chern-Simons model is defined on a sphere of radius R.

It is tricky to write down a matrix model with matrices assuming values on a 2 sphere. We by-pass this problem by using a specific coordinate system covering the sphere. Choose the sterographic projection coordinates y^i on the unit sphere with a metric

$$ds^2 = \frac{4(dy^i)^2}{(1 + y^2)^2}\,, \tag{3.11}$$

with the symplectic form

$$\frac{4dy^1 \wedge dy^2}{(1 + y^2)^2}\,. \tag{3.12}$$

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Integrating over the sphere we obtain the area of unit sphere 4π. For a pair of given eigen-values, we expect the Chern-Simons to yield a coupling

$$n\int dtf(y^2)\epsilon^{ij}y^i\dot{y}^j\,, \tag{3.13}$$

where $n\sim R^2/l_p^2$, the factor R^2 sets in since we are working with a sphere of radius R, and the factor l_p, as we just argued, is due to the fact that we are talking about planckian particles. The action (3.13) corresponds a symplectic form

$$n(2y^2f'+2f)dy^1\wedge dy^2\,. \tag{3.14}$$

Apart from the coefficient n, we expect the above to agree with (3.12), so f is solved to be

$$f(y^2)=-\frac{2}{y^2(1+y^2)}\,. \tag{3.15}$$

Semi-classically, the quantization of the Chern-Simons coupling (3.13) results in a Hilbert space of dimension

$$\frac{1}{2\pi}n\int\frac{4d^2y}{(1+y^2)^2}=2n\,, \tag{3.16}$$

so n is either an integer or half integer at the semi-classical level. The actual value of n is a little more complicated as we shall see shortly.

We need now to promote y^i to hermitean matrices Y^i, and to postulate a Chern Simons matrix action

$$S=\int dt\left(-2n\,\mathrm{tr}\,\frac{1}{Y^2(1+Y^2)}\epsilon^{ij}Y^i\dot{Y}^j-V(Y)\right). \tag{3.17}$$

As usual in matrix models defined on a curved space, there is an ordering ambiguity in the above action. We postpone discussing this issue to a later occasion. The potential term $V(Y)$ can not be specified by our limited knowledge of the microscopic theory. It is supposed to be rotationally invariant, and assume minimum when the two matrices Y^i commute. One can also contemplate the possibility of adding more matrices such as some fermionic matrices to the above action, to get a realistic model. We will not discuss this possibility in this paper.

It remains to discuss the exact quantization procedure and construct the Hilbert space for single eigen-values. To do so, it is convenient to switch to a more physical coordinate system. One choice is the cartesian coordinates for the sphere x^i, $i=1,2,3$, $\sum(x^i)^2=R^2$. The relation to the sterographic projection coordinates is

$$x^i=\frac{2R}{1+y^2}y^i\,,\qquad i=1,2\,. \tag{3.18}$$

The corresponding symplectic form

$$\frac{n}{R\sqrt{R^2-x^2}}dx^1\wedge dx^2\,, \tag{3.19}$$

where $x^2=(x^1)^2+(x^2)^2$. The Chern-Simons coupling is

$$-\frac{n}{R}\int dt\frac{\sqrt{R^2-x^2}}{x^2}\epsilon^{ij}x^i\dot{x}^j\,. \tag{3.20}$$

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This is the action of a unit charged particle moving in a monopole background with a magnetic charge n. The above action can be generalized to a nonrelativistic charged particle moving in the full 3 dimensional space

$$S = \frac{m}{2}\int \dot{x}^2 dt + \int A_i \dot{x}^i dt\,, \tag{3.21}$$

where the second term is to be identified with (3.20). It is well-known that the angular momentum of the system is corrected by a magnetic term

$$M^i = \epsilon^{ijk}(mx^j\dot{x}^k - mx^k\dot{x}^j) - \frac{nx^i}{r}\,. \tag{3.22}$$

Upon quantizing the action (3.21), one gets nontrivial commutators among $\dot{x}^i$, while the commutators among x^i vanish. For instance

$$[m\dot{x}^i, m\dot{x}^j] = i\epsilon^{ijk}B_k\,, \tag{3.23}$$

where B is the magnetic field generated by the monopole. One can check that the angular momentum components satisfy the usual $su(2)$ algebra

$$[M^i, M^j] = i\epsilon^{ijk}M^k\,. \tag{3.24}$$

Now taking the limit $m \to 0$, the first kinetic term in (3.21) drops out, thus the canonical momentum is not independent of x^i. The first term in (3.22) also drops out, the net angular momentum has a contribution solely from the magnetic field. In this limit, the quantization can be carried out by Dirac brackets. We shall not do this, we shall just put the system on a sphere of radius R and quantize the action (3.20). With the symplectic form (3.19), we find

$$[x^1, x^2] = -i\frac{R\sqrt{R^2 - x^2}}{n}\,. \tag{3.25}$$

It is straightforward to check that, to the first order, the commutation relations (3.24) still hold, with

$$M^i = -\frac{nx^i}{R}\,. \tag{3.26}$$

In fact, to make the quantization procedure rigorous, we simply assume the above relations. Thus x^i up to a constant form a $su(2)$ algebra. The Hilbert space is specified by picking an irreducible representation of spin J. Since the Casimir is

$$\sum_i (M^i)^2 = J(J+1)$$

we have

$$\sum_i (x^i)^2 = \frac{J(J+1)}{n^2}R^2\,. \tag{3.27}$$

Now x^i represent a fuzzy sphere. If we demand the radius of this fuzzy sphere to be exactly R, then

$$n^2 = J(J+1)\,. \tag{3.28}$$

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The dimension of the Hilbert space is $2J+1$. In the large J limit, $n \sim J$, so the dimension of the Hilbert space is approximately $2n$, a result agreeing with the previous semi-classical analysis. The horizon as a fuzzy sphere was discussed previously in [6].

It should be noted that we are not entitled to assume the radius of the fuzzy sphere to be R, this is because we are talking about the stretched horizon whose radius is not yet determined. This question is equivalent to determining n, and further microscopic knowledge is required in order to do so.

It is clear that what we have from the simple Chern-Simons model is giant gravitons. The cause of this phenomenon in our opinion is the vacuum energy whose nature is still mysterious, while giant gravitons in the AdS/CFT correspondence arise due to the R-R flux background [15]. Consequently, the mechanism to obtain the fuzzy sphere is different from the one in that case [16].

As we shall see in section 5 and section 6, it is even better to define Chern-Simons matrix model using a light-cone time. Moreover, the sterographic coordinates will have a direct physical meaning. In this language, it is possible to relate our approach to the approach in [4].

We have not gauged our matrix model. It is possible to introduce a gauge matrix A_0, and turn the time derivative in the Chern-Simons action (3.17) into a covariant derivative. This model becomes closer to the planar matrix model of [17]. Of course there is important difference: We have postulated that the potential $V(Y)$ is so chosen to force matrices Y^i become commuting for lowest energy.

According to discussions in the next section, it appears necessary to introduce fermionic matrices and perhaps other bosonic matrices. Once fermionic matrices are present, it is possible to introduce supersymmetry. SUSY is likely broken for most of the ground states, since in these states, the eigenvalues of Y^i are not homogeneously distributed. It is therefore very interesting to determine the SUSY breaking scale dynamically. We will come back to this important issue in future.

4. The membrane paradigm and holography

The matrix model proposed in the last section can be regarded as a concrete realization of the stretched horizon. As in the membrane paradigm [14], we should be able to assign various physical properties to the stretched membrane. For instance, consider that there is an electromagnetic field in the bulk enclosed by the horizon, the bulk electrodynamics should be completely specified by the electromagnetic properties of the stretched horizon, the conductivity of the horizon thus Ohm's law is a simple example.

More precisely, the boundary conditions for the electric field and the magnetic field on the stretched horizon are determined by the charge density and the current density on the stretched horizon, for instance

$$E_\perp = 4\pi\sigma\,, \qquad B_\parallel = 4\pi(j \times n)\,, \tag{4.1}$$

where σ is the charge density on the sphere, and j is the current density, n is the unit vector normal to the sphere.

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It should be possible to define charge and current density in the Chern-Simons matrix model if one assigns specific electro-magnetic properties to the excitations in the model. One possibility is to identify our model with some version of quantum Hall system in which all these quantities are clearly defined. In fact recently some planar matrix models are proposed to describe the real quantum Hall system [17]. One even can go a step further to incorporate the standard model fields in the bulk, by adding more matrices to the model to construct corresponding sources for these fields. However, it seems premature to construct even the charge density for the time being, since this is possible only when more details of the matrix model are given.

It is also important to understand how holography is realized in de Sitter space. Following the philosophy of AdS/CFT correspondence, one need to spell out a dictionary between operators in the matrix models and bulk modes. It seems straightforward to do this once more details are known about the matrix model. For example, to operators of the type

$$\mathrm{tr}\, Y^{i_1} \cdots Y^{i_n}\,, \tag{4.2}$$

one may associate spherical harmonic modes of a scalar field in the bulk. Apparently, in reality there are many fields in the bulk, we need to introduce more matrices in the matrix model in order to account for all of them along this line.

Whether it is really possible to generalize the AdS/CFT correspondence to the present context remains to be seen. An apparent objection to this naive generalization is that unlike there is no scaling argument available. One potential argument replacing the scaling argument is the infinite boost close to the horizon. Response of a given bulk mode under the boost may be translated to a certain property of the operator in the matrix model. We shall see in section 6 that matrix model can be built in another set of coordinates, the conformal light-cone coordinates, one may find out scaling argument in that context, for instance, the evolution with the light-cone time is related to dilation on the infinite past Euclidean space. Thus, the scaling dimension can be identified with the eigen-value of the matrix hamiltonian $H(Y) = V(Y)$:

$$[H(Y), \mathcal{O}] = h\mathcal{O}\,, \tag{4.3}$$

now the hamiltonian is taken to be the one evolving everything along the light-cone time, for more details, see section 6.

5. Some observations on boost transformations

Given the success of matrix theory in Minkowski space, one wonders whether it is possible to go to an infinite momentum frame in de Sitter space and to imitate matrix theory to get a smaller symmetry and write down a matrix model. Although the observations here are somewhat interesting, we are led to negative results.

We will be working with the steady-state metric

$$ds^2 = -dt^2 + e^{2t/R} dx_i^2\,. \tag{5.1}$$

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The apparent Killing vectors are those corresponding to translations in spatial directions and a shift in time together with rescaling of spatial coordinates

$$\begin{aligned} P_i &= \partial_i \,, \\ H &= \partial_t - \frac{x^i}{R}\partial_i \,. \end{aligned} \tag{5.2}$$

The conserved quantity associated with H is the AD energy [18]. For a massive particle of the action

$$S = -m \int dt \left(1 - e^{2t/R}(\dot{x}^i)^2\right)^{1/2} , \tag{5.3}$$

the AD energy is given by

$$E = m \left(1 - e^{2t/R}(\dot{x}^i)^2\right)^{-1/2} \left[1 + \frac{x^i \dot{x}^i}{R} e^{2t/R}\right] = (m^2 + e^{-2t/R}p^2)^{1/2} + \frac{x^i}{R}p^i \,. \tag{5.4}$$

If restricted inside the horizon

$$e^{t/R}|x| \leq R$$

the energy (5.4) is always positive.

Although it is possible to use the energy (5.4) to generate a flow, it is easy to see that the usual Hamilton-Jacobi equations are not valid, so one can not use (5.4) as a hamiltonian in the usual sense. Moreover, H does not commute with P_i.

The remaining Killing vectors are rotations in space and boost transformations. To construct a boost in a direction, say $z = x^{D-1}$, we go the the light-cone frame in which

$$ds^2 = \frac{4R^2}{(x^+ + x^-)^2}\left(-dx^+ dx^- + dx_\perp^2\right), \tag{5.5}$$

where the light-cone coordinates are

$$x^\pm = -Re^{-t/R} \pm z \,, \tag{5.6}$$

and the remaining coordinates $x_\perp$ are transverse directions. The metric (5.5) is conformal to the Minkowski metric, and the horizon for an observer at $x^i = 0$ is given by

$$x^+ x^- - x_\perp^2 = 0 \,. \tag{5.7}$$

The inside of the horizon is specified by the conditions $x^+ < 0$, $x^- < 0$ together with $x^+x^- - x_\perp^2 > 0$.

The boost transformation generalizing the flat space boost $x^\pm \to \exp(\pm\beta)x^\pm$ is rather involved, since the transformation acts on the transverse direction too. For dS_2, it is relatively easy to find out the boost transformation which is

$$x^\pm = \frac{\cosh\frac{\beta}{2}y^\pm \pm \sinh\frac{\beta}{2}R}{\cosh\frac{\beta}{2} \pm \sinh\frac{\beta}{2}y^\pm/R} \,. \tag{5.8}$$

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For small β, the above transformation reduces to the one in the flat space, where for small t, $x^{\pm}$ goes over to $t \pm z - R$. For large β, the nonlinear effect becomes important, and both $x^{\pm}$ goes over to constants $\pm R$. This is easy to understand for x^-, since $t - z$ shrinks to zero for large β, so x^- tends to $-R$. This limit is unfamiliar for x^+, since in the flat case one expects x^+ blow up instead of going over to a constant. The strange limit of the large boosts tells us that it is impossible to go to an infinite momentum frame and meanwhile to restrict everything within the horizon, because $x^+ \to R$ and is no longer negative.

For higher dimensional de Sitter space, the boost transformation is much more involved:

$$\begin{aligned} x^{\pm} &= \frac{\cosh\frac{\beta}{2}y^{\pm} \pm \sinh\frac{\beta}{2}R \pm \sinh\frac{\beta}{2}(fR)^{-1}y_{\perp}^2}{\cosh\frac{\beta}{2} \pm \sinh\frac{\beta}{2}y^{\pm}/R}\,, \\ x_{\perp} &= f^{-1}y_{\perp}\,, \end{aligned} \tag{5.9}$$

where

$$f = \left(\cosh\frac{\beta}{2} + \frac{1}{R}\sinh\frac{\beta}{2}y^{+}\right)\left(\cosh\frac{\beta}{2} - \frac{1}{R}\sinh\frac{\beta}{2}y^{-}\right) + \frac{1}{2R^2}(\cosh\beta - 1)y_{\perp}^2\,. \tag{5.10}$$

Although the analysis is complicated, the conclusion drawn in the previous paragraph still holds, namely it is impossible to go to the infinite momentum frame again.

Although our results are rather undesired, we can still ask whether there are corresponding longitudinal momenta which are conserved and scale under a boost just as in a flat space. The answer to this question is positive. Take dS_2 as an example. The Killing vector associated to boosting in the z direction can be obtained from (5.8) by expanding in β to the first order,

$$K = \frac{R}{2}\left[1 - \left(\frac{x^+}{R}\right)^2\right]\partial_+ - \frac{R}{2}\left[1 - \left(\frac{x^-}{2}\right)^2\right]\partial_-\,. \tag{5.11}$$

The algebra formed by this vector with the two Killing vector in (5.2) is

$$\begin{aligned} [E, P] &= \frac{1}{R}P\,, \qquad [P, K] = E\,, \\ [E, K] &= P - \frac{1}{R}K\,. \end{aligned} \tag{5.12}$$

This is of course the de Sitter algebra $\mathrm{so}(2,1) = \mathrm{sl}(2,\mathbb{R})$. Now the conventional longitudinal momenta $P^{\pm} = E \pm P$ are no longer eigenstates of K, rather, the following modified longitudinal momenta are

$$P^{\pm} = E \pm P \mp \frac{1}{R}K\,, \tag{5.13}$$

it is rather interesting to see that these conserved quantities involve the boost generator K. Under a finite boost of rapidity β, we have

$$P^{\pm} \to e^{\pm\beta}P^{\pm}\,. \tag{5.14}$$

One might wonder whether one can use the above boost property to simplify the kinetics of a particle. Again the answer is negative, since the on-shell condition in terms

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of $P^{\pm}$ is complicated, this is because, in terms of the conserved momentum P and energy E, the boost generator is not a simple function and involves coordinates directly.

Although the usual infinite momentum frame is impossible, it is still possible to implement the idea of the stretched horizon in the coordinate system (5.5). Take x^+ as the time variable. For fixed x^+, the horizon is located at

$$x^- = \frac{x_\perp^2}{x^+}\,. \tag{5.15}$$

The metric on the horizon is

$$ds^2_{\rm hor} = \frac{4R^2(x^+)^2}{((x^+)^2 + x_\perp^2)^2} dx_\perp^2\,. \tag{5.16}$$

This is the metric on a sphere of radius R, since the dependence on x^+ in the metric can be rescaled away. Now the coordinates $x_\perp$ provide a physical way to realize the sterographic projection coordinates we introduced in section 3 purely for convenience. Thus it may be more natural to study the Chern-Simons matrix model in which x^+ is taken as time, and the matrices assume values in the transverse space. It is even possible to make our intuitive argument for the Chern-Simons matrix model more concrete and appealing in the light-cone coordinates. We shall argue in the next section that it is better to take $\ln x^+$ as time in a theory on the horizon.

6. Relation to Strominger's approach

Strominger very recently proposed a dS/CFT correspondence [4]. In his approach, there is Euclidean CFT living on the past sphere at $\tau = -\infty$ in the global coordinates (3.3). The microscopic details of the CFT can not be specified except its central charge. The CFT is nonunitary, there are complex conformal weights.

In case of dS_3, the CFT is two dimensional. It is suggested that states in the Hilbert space are given in the radial quantization. Interestingly, it is shown in [4] that one can identify the AD Killing vector H in (5.2) with the radial generator $L_0 + \bar{L}_0$. Thus, in order to establish any connection between a horizon theory such as the spherical Chern-Simons matrix model in section 3 with the Euclidean conformal theory at the infinite past, it is important to identify this Killing vector in the theory on the horizon.

The AD Killing vector, up to a constant, can be written in the light-cone coordinates as

$$H = -(x^+\partial_+ + x^-\partial_- + x^i\partial_i)\,, \tag{6.1}$$

where we omitted the subscript $\perp$ for the transverse coordinates. Now we ask, taking x^+ as our time variable, how the AD Killing vector acts on a function on the horizon. Substitute the constraint (5.15) into the function $f(x^+, x^-, x^i)$, we obtain a function

$$\tilde{f}(x^+, x^i) = f\left(x^+, \frac{x^2}{x^+}, x^i\right). \tag{6.2}$$

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We find that the operators acting on the function $\tilde{f}$ are related to operators acting on f in the following way

$$x^+\partial_+ \to x^+\left(\partial_+ + \frac{x^2}{(x^+)^2}\partial_-\right),$$
$$x^i\partial_i \to x^i\left(\partial_i - \frac{2x^i}{x^+}\partial_-\right). \tag{6.3}$$

Using these relations we see that when acting function $\tilde{f}$, the AD Killing vector reduces to

$$H = -(x^+\partial_+ + x^i\partial_i). \tag{6.4}$$

This reduction is to be expected, since the Killing vector in (6.1) is the dilation operator for all coordinates including x^-. Now the constraint (5.15) respects the dilation, so when restricted to the horizon, the dilation operator should be (6.4).

This AD Killing vector on the horizon can be further simplified by noticing that the metric restricted on the horizon depends on x^+, as in (5.16). Performing the rescaling $x^i \to x^i x^+$, the metric becomes

$$ds^2_{\rm hor} = \frac{4R^2}{(1+x^2)^2}dx^2\,. \tag{6.5}$$

Namely, x^i are just the sterographic coordinates y^i introduced in section 3. By a similar argument as the above, we find that now in terms of x^+ and the sterographic coordinates x^i, the AD Killing vector is simply

$$H = -x^+\partial_+\,. \tag{6.6}$$

Now it is clear that, given the fact that H should be a conserved quantity, we should take $-\ln(-x^+)$ as the time for a theory on the horizon. We use $-x^+$ in the logarithmic since inside the horizon $x^+ < 0$. The light-cone time $\tau = -\ln(-x^+)$ increases with x^+ and tends to ∞ as $x^+ \to 0$, its range is $(-\infty, \infty)$.

So it is natural to build up the matrix Chern-Simons model introduced in section 3 directly on the horizon in the light-cone coordinates. The matrices Y^i just correspond to the transverse coordinates discussed in the previous section and the present section. Furthermore, the matrix hamiltonian is identified with the AD Killing vector on the horizon, thus should be identified with the radial generator in the Euclidean conformal theory of Strominger.

Just as Strominger, here we can not say much about microscopic details of the hamiltonian in the Chern-Simons matrix model, which is simply given by the matrix potential $V(Y)$ in (3.17). Although it is just a function of matrices Y^i, its dynamic content is nontrivial since entries of matrices Y^i are noncommutative.

7. Discussions

Quantum gravity on de Sitter spaces is not expected to be found in any time soon, not only because there are conceptual challenges, but also because if the fate of our universe is such

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a space, there are many realistic questions to answer, such as supersymmetry breaking and fitting the standard model parameters. Nonetheless these universes provide an excellent place to test our ideas about string/M theory and will help in the long term search for fundamental principles and underlying degrees of freedom.

We present one possible holographic theory for 4 dimensional de Sitter space. According to our experience with matrix theory, the matrix model is highly dimension-dependent, so we do not attempt to say anything about de Sitter spaces in other dimensions. In our opinion, even if our model will ultimately be proven to be on a wrong track, it is useful for stimulating other ideas. Further, as we tried in section 6, it will be useful to compare our attempt to other attempts now available or to appear in future.

It is interesting to compare the matrix model of the de Sitter space with the matrix description of Schwarzschild black holes in a flat space [19]. Both descriptions are a version of the stretched horizon picture. Nevertheless there are fundamental differences. The first obvious distinction is that the de Sitter matrix model describes the interior physics viewed by an observer inside the horizon, while the matrix black hole describes the black hole as seen by an outside observer, presumably encoding information about the whole black hole including its interior, although one is not supposed to talk about the causally disconnected interior for the outside observer. Secondly, the de Sitter matrix model is the whole theory of the de Sitter space, that is, its Hilbert space seen by one observer is the whole Hilbert space of the de Sitter space, according to the cosmological complementarity principle. The matrix black hole only describes a subset of excited states within matrix theory, and itself can not be viewed as a complete, unitary theory. In fact the black hole states are only metastable. The most conspicuous difference is that the de Sitter space matrix model should not only describe the pure de Sitter space, but also other states which are asymptotically de Sitter. Thus, it should also describe a state where there are a cosmic horizon and a black hole horizon, although at present it is not clear how to construct such a state in the matrix model.

The cosmological initial conditions can in principle be discussed in our matrix model. When nonabelian matrices are sufficiently excited, the dynamics does not necessarily describe physics only near the cosmological horizon, since there is no clear geometric picture of horizon in this situation.

A difficult question is how to realize cosmological complementarity principle, namely how to relate the Hilbert for one observer to the one for another observer. Perhaps a bulk formulation is needed to order to establish a bridge. It is not a far-off speculation that the bulk theory is a theory realizing spacetime noncommutativity and holography along the line of [20].

One may speculate a lot more about things related to our approach, but it is safe to leave many interesting issues for the future.

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